Line source distributions and slender-body theory

By JOHN P. MORAN
Therm Advanced Research, Ithaca, New York

(Received 18 December 1962 and in revised form 2 May 1963)

A systematic procedure is presented for the determination of uniformly valid successive approximations to the axisymmetric incompressible potential flow about elongated bodies of revolution meeting certain shape requirements. The presence of external disturbances moving with respect to the body under study is admitted. The accuracy of the procedure and its extension beyond the scope of the present study—e.g. to problems in plane flow—are discussed.

1. Introduction

Slender-body theory is one of the most useful analytical tools available to fluid dynamicists, since it yields closed-form approximate solutions to many problems of practical importance. The theory originated in Munk's (1924) analysis of the lateral flow past elongated bodies of revolution. Munk postulated that the flow past any cross-section is approximately independent of that past any other, and used two-dimensional theory to determine the cross-flow at each station. Later, von Kármán (1927) treated the same problem by distributing along the body axis doublets oriented normal to the axis. Still later, Munk (1934) showed the two approaches to be equivalent for slender bodies to a first approximation.

The first to apply slender-body theory to axisymmetric flow problems was Weinig (1928), who employed axial distributions of doublets aligned with the axis. Munk (1934), using axial source distributions, obtained equivalent formulas.

Subsequently, second-order theories (Van Dyke 1954, 1959) and other modifications were introduced to improve the accuracy of the slender-body theory. However, these also depend on the use of axial singularity distributions. Although Vandrey (1951) and Landweber (1951, 1959) were able to simplify the surface-singularity-distribution approach somewhat by using slender-body-type approximations, their resultant iteration procedures are still too complex to yield results in closed form.

It thus appears that the analytical virtues of slender-body theory are inextricably tied up with the use of axial singularity distributions. The present attempt to approximate the axisymmetric flow past elongated bodies of revolution is, therefore, based on the distribution of sources along the body axis.

The study is restricted to bodies whose ends are parabolic (blunt with finite radius of curvature), and whose cross-sectional area distribution is expandable about the stagnation points in power series which converge over the entire length of the body. It has been shown (Moran 1961) that line source distributions are most likely to yield closed-form results under these restrictions.

The integral equation governing the source strength is inverted by a technique based on that developed by Landweber (1951). Thus, in §§ 3.1 and 3.2, separate successive-approximation procedures are set up for determining sequentially the extent and the form of the source distribution.

These procedures are applied in § 3.3 to obtain a general formula for the second approximation to the source strength. Use of this formula yields an approximation to the flow field about bodies of the type described above which is uniformly valid to second order in the perturbation parameter τ^2, τ being the body thickness ratio. This establishes the applicability of axial source distributions to such bodies, in the sense that the technique yields at least an asymptotic expansion of the exact solution powers of τ^2. Unfortunately, the ability of our analysis to yield terms in this expansion beyond the second term, without further restrictions on the body shape, is not rigorously established, nor is the convergence of the procedure.

Our second approximation for the source strength differs from the corresponding result of formal slender-body theory (Van Dyke 1959) only in the extent of the distribution. In the formal theory, the sources extend to the ends of the body, thus inducing spurious singularities at the stagnation points. We instead introduce gaps between the ends of the distribution and the stagnation points, and take care to determine the correct extent of the gaps (§ 3.1). This suggests a technique for rendering formal slender-body theory uniformly valid, which may be applicable outside the scope of the present research. In problems where the formal solution consists of singularities distributed along some mean surface or line, one may be able to construct a uniformly valid solution simply by predetermining the proper extent of the distribution with a method like that of § 3.1.

From comparisons with exact solutions, it is shown in § 4.3 that the second approximation is sometimes sufficient for practical purposes. However, there also exist cases, within the class of body shapes studied, for which the successive approximations converge rather slowly, and perhaps only asymptotically as $\tau \to 0$. It is tentatively suggested that such cases may be identified by examining the expansions in τ^2 of the gaps between the ends of the source distribution and the ends of the body, the first five terms of which are given explicitly in § 3.1.

In § 4.4, the approach developed in §§ 3.1 and 3.2 for treating steady, unbounded flows about isolated bodies of revolution is extended to a class of unsteady interference problems, in which the presence of external singularities moving with respect to the body under study must be considered.

Many of the results presented in this paper were previously published in a Therm Advanced Research report (Moran 1962), to which we shall refer as I.

2. Formulation

2.1. *Scope of study*

Our subject is the axisymmetric flow of a perfect fluid about a slender body of revolution. We shall use body-fixed cylindrical co-ordinates (x, r) with origin at the body nose, as shown in figure 1. The flow far from the body is uniform, and is directed along the positive x-axis with speed U. Initially, it will be supposed that the body is isolated. This restriction will be removed in § 4.4 to permit the consideration of certain unsteady interference problems.

The body shape is defined by $r = R(x)$, and its length is set equal to unity. The maximum diameter of the body is then τ, the thickness ratio. We require that the ends of the body be blunt with finite radius of curvature, and that the

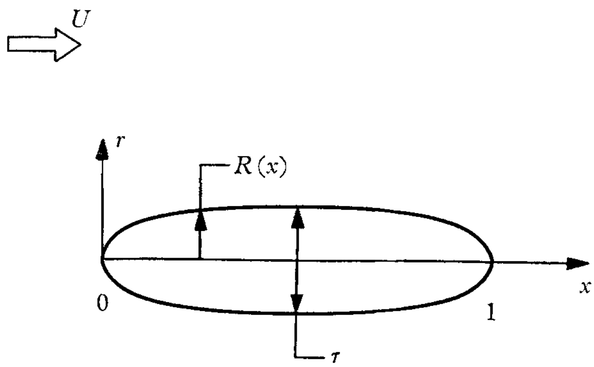

FIGURE 1. Co-ordinates and nomenclature.

cross-sectional area distribution $S(x)$ be expandable about the stagnation points in power series which converge over the entire body length

$$S(x) \equiv \pi R^2(x) = \pi \sum_1^\infty a_n x^n = \pi \sum_1^\infty b_n (1-x)^n. \tag{2.1}$$

For the series (2.1) to converge in the interval $0 \leqslant x \leqslant 1$, the coefficients a_n and b_n must all be of the same order of magnitude. Since R^2 is of order τ^2, we therefore require

$$a_n, b_n = O(\tau^2). \tag{2.2}*$$

We further stipulate that the leading coefficients of the series (2.1) are not zero or of order τ^4, although such conditions would satisfy (2.2); a_1 and b_1 may be neither small nor large compared with τ^2. Since a_1 and b_1 are twice the radii of curvature of the nose and tail, respectively, this stipulation simply strengthens our restriction that the ends of the body be blunt; the order of magnitude of the radius of curvature, which is a measure of the bluntness, is now required to be the maximum consistent with equation (2.2).

2.2. *Basic equations*

It is assumed that the flow is incompressible and irrotational. The problem is then conveniently formulated in terms of a velocity potential Φ, whose gradient yields the velocity field, and which gives the pressure field through Bernoulli's equation.

From continuity, Φ must satisfy Laplace's equation everywhere in the flow field outside the body. If the analytic continuation of the potential across the

* In the way of nomenclature, for any k, including zero, $y = O(\tau^k)$ means that as $\tau \to 0$, (y/τ^k) is less than some finite constant, while $y = o(\tau^k)$ means that $(y/\tau^k) \to 0$ as $\tau \to 0$.

body surface is free from singularities except on the axis, the potential may be written

$$\Phi(x, r) = Ux - \frac{U}{4\pi}\int_\alpha^\beta \frac{f(\xi)\,d\xi}{\{(x-\xi)^2+r^2\}^{\frac{1}{2}}}. \tag{2.3}$$

The first term in (2.3) is the potential of a uniform flow in the x-direction, while the second is the potential of a distribution of sources along the x-axis in the interval (α, β). The quantity $Uf(x)$ is the strength per unit length of the distribution, the strength of a source being defined as the volume rate of flow across any surface enclosing the source.

Since the velocity due to a source is infinite at the source, the distribution must be confined within the body, and there must generally be finite gaps between the ends of the distribution and the stagnation points; i.e. $\alpha > 0$ and $\beta < 1$. Then the source-induced velocities vanish at infinity, and the potential (2.3) automatically satisfies the boundary condition of uniform flow at infinity.

The potential must also satisfy a flow-tangency condition on the body surface

$$\Phi_r - \Phi_x R'(x) = 0 \quad \text{on} \quad r = R(x), \tag{2.4}$$

where the subscripts indicate partial differentiation. Substituting in (2.4) for Φ from (2.3), we multiply the resultant expression by $R(x)$, and integrate over x. The constant of integration is evaluated at $x = 0$, using the fact that the net source strength associated with a closed body must be zero

$$\int_\alpha^\beta f(\xi)\,d\xi = 0. \tag{2.5}$$

From these manipulations we obtain the integral equation governing the source strength

$$2\pi R^2(x) = \int_\alpha^\beta f(\xi)\,K(x,\xi)\,d\xi, \tag{2.6}$$

where the kernel is

$$K(x,\xi) \equiv (x-\xi)\{(x-\xi)^2 + R^2(x)\}^{-\frac{1}{2}}. \tag{2.7}$$

Since the flow is axisymmetric we may also work in terms of the Stokes stream function Ψ, related to Φ by

$$\Phi_x = \frac{1}{r}\Psi_r, \quad \Phi_r = -\frac{1}{r}\Psi_x. \tag{2.8}$$

By definition, the stream function is constant on streamlines of the flow. The integral equation (2.6), in fact, simply says that on the body surface,

$$\Psi(x, R(x)) = 0. \tag{2.9}$$

2.3. *Outline of solution*

In inverting the integral equation (2.6) for the source strength, we shall follow Landweber (1951) by determining separately the extent and the functional form of the distribution. These two tasks require different approaches, which may be pursued sequentially. To find the gaps between the ends of the distribution and the stagnation points, we employ a successive-approximation procedure based on power-series expansions. Once the extent of the distribution is known, its form can be determined by iterative solution of the integral equation. Both procedures depend heavily on the assumed slenderness of the body, the results being expressed as expansions in even powers of the thickness ratio, τ. In the analysis, certain assumptions will be made regarding the form of the source distribution. The validity of these assumptions will be discussed in § 4.1.

3. Solution

3.1. *Determination of extent of distribution*

We expect that errors in the determination of the extent of the source distribution lead to violations of the body boundary condition which are most serious near the ends of the distribution. To emphasize these regions, the procedures for calculating the end-points of the distribution are based on power-series expansions of the integral equation (2.6) about the stagnation points. Thus, the equations governing α, the gap between the nose and the leading edge of the distribution, are obtained by expanding the kernel of (2.6) in a power series about the nose, substituting for R^2 its expansion (2.1) in x, and equating coefficients of like powers of x. The coefficients of x^0 simply reproduce equation (2.5), which is unimportant in the determination of α.* By equating the coefficients of the next six higher powers of x we obtain

$$a_1 = a_1 I_2, \tag{3.1}$$

$$0 = a_1 I_3 - \tfrac{3}{8} a_1^2 I_4, \tag{3.2}$$

$$0 = \tfrac{3}{8} a_1 (4 - a_2) I_4 - \tfrac{3}{2} a_1^2 I_5 + \tfrac{5}{16} a_1^3 I_6, \tag{3.3}$$

$$\begin{aligned} 0 = {} & \tfrac{3}{8}(4a_2 - a_2^2 - a_1 a_3) I_4 + a_1 (2 - 3a_2) I_5 \\ & - \tfrac{15}{16} a_1^2 (4 - a_2) I_6 + \tfrac{15}{8} a_1^3 I_7 - \tfrac{35}{128} a_1^4 I_8, \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 = {} & \tfrac{3}{8}(4a_3 - 2a_2 a_3 - a_1 a_4) I_4 + (2a_2 - \tfrac{3}{2} a_2^2 - 3a_1 a_3) I_5 \\ & + \tfrac{5}{16} a_1 (8 - 24a_2 + 3a_2^2 + 3a_1 a_3) I_6 - \tfrac{15}{8} a_1^2 (4 - 3a_2) I_7 \\ & + \tfrac{35}{32} a_1^3 (6 - a_2) I_8 - \tfrac{35}{16} a_1^4 I_9 + \tfrac{63}{256} a_1^5 I_{10}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} 0 = {} & \tfrac{3}{8}(4a_4 - a_3^2 - 2a_2 a_4 - a_1 a_5) I_4 + (2a_3 - 3a_2 a_3 - 3a_1 a_4) I_5 \\ & + \tfrac{5}{16}(8a_2 - 12a_2^2 - 24a_1 a_3 + a_2^3 + 3a_1^2 a_4 + 6a_1 a_2 a_3) I_6 \\ & + \tfrac{3}{8} a_1 (8 - 40a_2 + 15a_2^2 + 15a_1 a_3) I_7 \\ & - \tfrac{35}{64} a_1^2 (24 - 36a_2 + 3a_2^2 + 2a_1 a_3) I_8 + \tfrac{35}{4} a_1^3 (2 - a_2) I_9 \\ & - \tfrac{315}{256} a_1^4 (8 - a_2) I_{10} + \tfrac{315}{128} a_1^5 I_{11} - \tfrac{231}{1024} a_1^6 I_{12}, \end{aligned} \tag{3.6}$$

where we have defined the functional of the source strength

$$I_k \equiv \frac{1}{4\pi} \int_\alpha^\beta \xi^{-k} f(\xi)\, d\xi. \tag{3.7}$$

We now assume that the source strength f is expandable in power series about the stagnation points

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} d_n (1-x)^n. \tag{3.8}$$

* As we shall see below, the integral equation (2.6) need be satisfied only to order τ^2 for our present purposes, while the contribution of α to equation (2.5) is of order τ^4.

It is also assumed that the radii of convergence of these series are of order unity. Then all the coefficients c_n and d_n are of the same order in τ^2. Specifically, c_n and d_n are of order τ^2, since $f = O(\tau^2)$, as may be deduced by examination of the integral equation (2.6). As was the case with a_1 and b_1 in equation (2.1), we further assume that the leading coefficients of the series (3.8), c_0 and d_0, are neither small nor large compared with τ^2.

With these assumptions we are precluding the possibility of discontinuities in the source strength (or in any of its derivatives) and, in particular, the presence of discrete sources near the ends of the distribution. The assumptions are not so strong, however, that such discontinuities or discrete singularities are completely inadmissible; all that is required is that their distances from the ends of the body be of the order of a body length. That is, equations (3.8) need represent f only in regions near the stagnation points whose lengths are of order unity.

Now when $\xi = O(1)$, the integrand of the functional I_k defined in equation (3.7) is of order τ^2. Therefore, the contribution to I_k of that part of the source distribution which, in accordance with the above remarks, is not described by equation (3.8) is also of order τ^2. Further, anticipating that α and $1-\beta$ are small for slender bodies, we may simplify the contribution to equation (3.7) of that part of f which is described by (3.8) with the result

$$4\pi I_k = \frac{\alpha^{1-k}}{k-1}c_0 + \frac{\alpha^{2-k}}{k-2}c_1 + \ldots + \frac{1}{\alpha}c_{k-2} + o(\tau^0). \tag{3.9}$$

Using (3.9) we may now write equation (3.1) as

$$a_1 = \frac{1}{4\pi}\frac{a_1}{\alpha}c_0 + o(\tau^2). \tag{3.10}$$

Since, by assumption, a_1 and c_0 are neither small nor large compared with τ^2, equation (3.10) shows that α must behave similarly

$$\alpha = O(\tau^2), \quad \frac{1}{\alpha} = O\left(\frac{1}{\tau^2}\right). \tag{3.11}$$

More specific information on α is obtained from equation (3.2), which may be simplified with the help of (3.9) to

$$\frac{1}{2}\frac{a_1}{\alpha^2}\left[1 - \frac{1}{4}\frac{a_1}{\alpha}\right]c_0 + \frac{a_1}{\alpha}\left[1 - \frac{3}{16}\frac{a_1}{\alpha}\right]c_1 = o(\tau^2). \tag{3.12}$$

Since the second term on the left side of (3.12) is of order τ^2, so must be the first term, which implies that

$$a_1/\alpha = 4 + o(1). \tag{3.13}$$

To find a more accurate approximation for α we write

$$a_1/\alpha = 4 + B. \tag{3.14}$$

In the equation formed by substituting (3.9) and (3.14) into (3.3), the term involving c_0 appears to be of order unity. As in equation (3.12), this cannot be so, since the terms involving c_n ($n \neq 0$) are of order τ^2, while the sum of all the terms must be of order τ^4.* Thus requiring the term which involves c_0 to be of order τ^2 we determine B to order τ^2.

This same procedure was applied in succession to equations (3.4), (3.5), and (3.6), so that a_1/α was determined to a fifth approximation. Solving the resultant expression for α, we have

$$\begin{aligned}\alpha &= \tfrac{1}{4}a_1 - \tfrac{1}{16}a_1a_2 + \tfrac{1}{64}(a_1^2a_3 + 2a_1a_2^2) \\ &\quad - \tfrac{1}{256}(a_1^3a_4 + 7a_1^2a_2a_3 + 5a_1a_2^3) \\ &\quad + \tfrac{1}{1024}(a_1^4a_5 + 10a_1^3a_2a_4 + 6a_1^3a_3^2 + 37a_1^2a_2^2a_3 + 14a_1a_2^4) + o(\tau^{10}). \qquad (3.15)\end{aligned}$$

The determination of β proceeds along similar lines, with the various power-series expansions centred about $x = 1$. Because of the reversibility of axisymmetric incompressible irrotational flow, the form of the expression derived for $(1-\beta)$ is identical with that of equation (3.15). Thus

$$1-\beta = \tfrac{1}{4}b_1 - \tfrac{1}{16}b_1b_2 + \ldots, \qquad (3.16)$$

where the b_n's are the coefficients of the power-series expansion of R^2 about the tail, see equation (2.1).

The first to note the importance of correctly fixing the extent of the singularity distribution was Flügge-Lotz (1931), who suggested a procedure based on Taylor-series expansion of the governing integral equation about the stagnation points. This is essentially the approach devised independently by Landweber (1951) and followed here.

The most significant difference between the present procedure and that of Landweber is the choice between sources and doublets for the axial singularity distribution. Since each jump discontinuity in a distribution of doublets corresponds to the presence of a discrete source, such a distribution can, in principle, represent a wider class of bodies than can a piecewise-continuous axial source distribution. But as exemplified by the half-body created by a point source in a uniform flow (see Landweber 1951, for example), if a discrete source appears near the end of the distribution, the specification of the body shape is generally complicated by the fact that the power-series expansion of R^2 about the nose converges only in a limited region near the nose.

A further difference between Landweber's procedure and that employed here is our use of order-of-magnitude arguments, which avoids the necessity for truncating equations like (3.10) and (3.12) arbitrarily, and for finding α by solution of a determinantal equation. Nevertheless, had Landweber stipulated that the doublet strength vanish at the ends of the distribution, as would be necessary to preclude the presence of point sources there, his results would agree with ours. As it is, under restrictions on the body shape similar to those set forth here in § 2.1, he finds α correctly only to a first approximation (I). For practical purposes, however, Landweber's determination of α seems to be sufficiently accurate (see §§ 3.2 and 4.3 below). A similar comment applies to certain analyses

* Actually, of order $\tau^4 \ln \tau^2$. For brevity, this distinction will often be ignored in the text, but the equations will be made rigorous by using the small o notation.

(see I for bibliography) in which the source distribution was terminated at points midway between the stagnation points and the centres of curvature of the body's extremities. This choice, which agrees with equations (3.15) and (3.16) to a first approximation, was based on the known exact solution for the prolate ellipsoid of revolution.

3.2. *Determination of form of source distribution*

Again following Landweber, we now seek to determine the functional form of the source strength by iterative solution of the integral equation (2.6). We consider first the following generalization of that equation

$$G(x) = \int_\alpha^\beta g(\xi)\,(x-\xi)\,\{(x-\xi)^2+R^2(x)\}^{-\frac{1}{2}}\,d\xi. \tag{3.17}$$

It is convenient to differentiate (3.17) with respect to x

$$G'(x) = \int_\alpha^\beta g(\xi)\left[R^2(x)-\tfrac{1}{2}(x-\xi)\frac{dR^2}{dx}(x)\right][(x-\xi)^2+R^2(x)]^{-\frac{3}{2}}\,d\xi. \tag{3.18}$$

This facilitates an approximate solution for g suitable for slender bodies, since the integrand of (3.18) tends to peak near $\xi = x$ if R^2 is small, while the kernel of (3.17) is of order unity no matter how slender the body is.

We assume that the first two derivatives of g exist for $\alpha < x < \beta$, so that in that interval, from Taylor's expansion formula with remainder,

$$g(\xi) = g(x)+(\xi-x)\,g'(x)+\frac{(\xi-x)^2}{2}\,g''(x_1) \tag{3.19}$$

for some x_1 in (α, β). We also assume that, over the entire length of the distribution, g' and g'' are of the same order of magnitude in τ^2 as is g. Then, substituting for $g(\xi)$ in equation (3.18) from (3.19) we find

$$G'(x) = g(x)\left[\left\{\beta-x-\frac{1}{2}\frac{dR^2}{dx}(x)\right\}\{(\beta-x)^2+R^2(x)\}^{-\frac{1}{2}}\right.$$
$$\left.+\left\{x-\alpha+\frac{1}{2}\frac{dR^2}{dx}(x)\right\}\{(x-\alpha)^2+R^2(x)\}^{-\frac{1}{2}}\right]+o(g)^* \quad (\alpha < x < \beta). \tag{3.20}$$

Now when $x \gg \alpha$, it is clear that

$$\left\{x-\alpha+\frac{1}{2}\frac{dR^2}{dx}(x)\right\}\{(x-\alpha)^2+R^2(x)\}^{-\frac{1}{2}} = 1+O(\tau^2). \tag{3.21}$$

It may not be obvious that this relation also holds when x is small. To show this, we suppose that $x = O(\tau^2)$. Then, using equations (2.1) and (3.13), we may approximate the terms on the left side of (3.21) by

$$\frac{x-\frac{1}{4}a_1+\frac{1}{2}a_1+O(\tau^4)}{[(x-\frac{1}{4}a_1)^2+a_1x+O(\tau^6)]^{\frac{1}{2}}} = \frac{x+\frac{1}{4}a_1+O(\tau^4)}{[(x+\frac{1}{4}a_1)^2+O(\tau^6)]^{\frac{1}{2}}} \quad (x = O(\tau^2)). \tag{3.22}$$

* In identifying the remainder term as being $o(g)$—i.e. as being vanishingly small compared to g as $\tau \to 0$—the expansions in τ^2 developed in the Appendix for certain radicals and logarithmic terms are useful.

Thus equation (3.21) is valid for all x of interest. A similar relation holds for the other term in the square brackets of (3.20), which may then be written

$$G'(x) = g(x)\,[2+o(\tau^0)]. \tag{3.23}$$

Therefore, the approximate solution of equation (3.17) is

$$g(x) = \tfrac{1}{2}G'(x)\,[1+o(\tau^0)]. \tag{3.24}$$

This approximation is uniformly valid, in that the error is of the indicated order over the entire interval $\alpha < x < \beta$.

In order to apply equation (3.24) to the iterative solution of (2.6), we define an nth approximation to the source strength, $f_n(x)$, such that

$$f(x)-f_n(x) = o(\tau^{2n}) \tag{3.25}$$

over the entire length of the distribution. Since $f = O(\tau^2)$ it is consistent to define $f_0 = 0$.

Suppose that f_{n-1} is known. Since the kernel and integration interval of equation (2.6) are both of order unity, we may write that equation as

$$2S(x) - \int_\alpha^\beta f_{n-1}(\xi)\,K(x,\xi)\,d\xi + o(\tau^{2n}) = \int_\alpha^\beta [f_n(\xi)-f_{n-1}(\xi)]\,K(x,\xi)\,d\xi. \tag{3.26}$$

From equation (3.25), the right side of (3.26) is of order τ^{2n}. Comparing equations (3.17) and (3.26), we then use (3.24) to write the solution of (3.26) as

$$f_n(x) = f_{n-1}(x) + S'(x) - \frac{1}{2}\int_\alpha^\beta f_{n-1}(\xi)\,K_x(x,\xi)\,d\xi, \tag{3.27}$$

where, as is permitted by (3.25), we have ignored terms of order τ^{2n+2}.

Setting $f_0 = 0$ in (3.27), we find the first approximation to f,

$$f_1(x) = S'(x). \tag{3.28}$$

It is noteworthy that equation (3.28) is identical with the familiar result of formal slender-body theory. Substituting (3.28) into (3.27), we may determine the second approximation to the source strength, f_2, and so on *ad infinitum.*

The first attempt to solve the axial-singularity-distribution problem by iteration on the governing integral equation was made by Weinig (1928). However, his successive approximations diverged, apparently because he allowed the singularity distribution to extend to the ends of the body (I).

Landweber's (1951) analysis does not, of course, suffer from this defect. The equation he derived for the iterative calculation of the doublet strength is identical in form with our equation (3.27). However, it is not inherently as accurate. In its derivation, terms similar to those in the square brackets of our equation (3.20) must be approximated by constants, which approximation can be shown to be invalid near the ends of the distribution (contrast our equations (3.21) and (3.22)).

In particular, Landweber's first approximation for the doublet strength is simply the slender-body result, $S(x)$, which allows the strength to be nonvanishing at the ends of the distribution. Actually, Landweber predetermined the doublet strength at the ends of the distribution with a series-expansion procedure, and modified his iteration scheme accordingly. However, it can be shown (I) that Landweber's formula for the terminal value of the doublet strength is accurate only to first order, as is his determination of the extent of the singularity distribution (§ 3.1). Strictly speaking, then, Landweber's iterative solution is uniformly valid only to a first approximation. But since his iterations are intended to be carried out numerically, and since the errors of his first approximations for the distribution extent and for the terminal values of the doublet strength are extremely small numerically, Landweber's procedure yields quite satisfactory results for such quantities of interest as the body surface pressure distribution (see § 4.3).

3.3. *Second-order source strength*

It is convenient at this point to apply the iteration procedure described above to determine the second approximation to the source strength. According to equations (3.27) and (3.28),

$$f_2(x) = 2S'(x) + J'(x), \tag{3.29}$$

where

$$J(x) \equiv -\frac{1}{2}\int_\alpha^\beta S'(\xi)\,K(x,\xi)\,d\xi. \tag{3.30}$$

In view of the restrictions on the body shape set forth in § 2.1, $S'(\xi)$ is analytic, and so may be expanded in a Taylor series about x. Then,

$$J(x) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n!}S^{(n+1)}(x)\,H_{n+1}(x), \tag{3.31}$$

where $S^{(k)}(x)$ denotes the kth derivative of S with respect to x, and

$$H_k(x) \equiv \int_\alpha^\beta \frac{(\xi-x)^k}{\{(\xi-x)^2+R^2(x)\}^{\frac{1}{2}}}\,d\xi. \tag{3.32}$$

A straightforward evaluation of the integrals H_k would lead to the inclusion in f_2 of terms of order τ^6 and higher. The simplifications of these integrals required to avoid these extraneous terms are outlined in the Appendix. Using the results derived therein, we find the following general formula for the second approximation to the source strength

$$f_2(x) = S'(x) - \frac{1}{4\pi}\frac{d}{dx}\left\{\left[\frac{S(x)}{x}\right]^2 + \left[\frac{S(x)}{1-x}\right]^2 + S(x)\,S''(x)\ln\left[\frac{4\pi x(1-x)}{S(x)}\right] + 2S(x)\sum_{n=1}^{\infty}\frac{n+1}{n(n+2)!}h_n(x)\,S^{(n+2)}(x)\right\}, \tag{3.33}$$

where

$$h_k(x) \equiv (1-x)^k + (-x)^k. \tag{3.34}$$

4. Discussion

4.1. *Validity of assumptions*

In principle, the source strength can be determined to arbitrarily high order in τ^2 by repeated application of equation (3.27). Unfortunately, the requisite integrations soon become almost hopelessly complex. We have not even been able to derive a general formula for f_3 in closed form, most of the difficulty being due to the presence of the logarithmic term in the result for f_2, equation (3.33).

Our lack of a closed form result for f_n for n arbitrarily large precludes a discussion of the convergence of our procedure. Thus we must admit the possibility that, for some bodies within the class defined in § 2.1, the successive approximations yield only an asymptotic expansion of the solution in τ^2. Moreover, we are unable to verify the several assumptions made in the derivations of §§ 3.1 and 3.2 on the behaviour of the source strength. Thus these assumptions may imply restrictions on the shapes of bodies to which our procedures are applicable which are more severe than those delineated in § 2.1.

On the other hand, it is not difficult to show that, within the restrictions of § 2.1, our second approximation to f does behave as assumed. The first-order part of equation (3.33) is analytic all along the body axis, and all its derivatives are everywhere of order τ^2. The second-order part is also analytic for $0 \leqslant x \leqslant 1$, while all its derivatives are of order τ^4 in this interval. Also, the leading terms of the power-series expansions of f_2 about the stagnation points are neither small nor large compared to τ^2, and the radii of convergence of these series are of order unity.

Let us then consider the function $\Phi_2(x, r)$ (say) formed by substituting in equation (2.3) for α, β, and f their second approximations from equations (3.15), (3.16), and (3.33), respectively. This function exactly satisfies the Laplace equation and the boundary condition at infinity. Also, for bodies shaped as described in § 2.1, the error term obtained when Φ_2 is substituted into the body boundary condition (2.4) is of order τ^4 smaller than that of the zeroth approximation to Φ (no sources) over the entire length of the body. To show this, note that the product of (2.4) and R is the x-derivative of the integral equation (2.6), and use equations (3.17)–(3.23). Thus our second approximation to the source strength yields an approximation to the potential which is uniformly valid to second order in τ^2 throughout the flow field about a body of the type considered, *regardless of whether or not our procedures are capable of yielding higher approximations.*

This establishes the applicability of axial source distributions to bodies meeting the restrictions of § 2.1, in the sense that the technique yields at least an asymptotic expansion of the exact solution in even powers of the thickness ratio. To be sure, we can claim only to have found the first two terms of the expansion. But since there is no reason to believe that the behaviour of higher approximations to f differs from that of f_2, it seems plausible that our procedure could yield higher-order terms of the expansion as well.

4.2. *On rendering slender-body theory uniformly valid*

The starting point of the usual derivation of slender-body theory is the small-disturbance assumption that the perturbation velocity components $\Phi_x - U$ and Φ_r are everywhere small compared to the speed U of the oncoming flow. Because this assumption is invalid near stagnation points, a theory based thereon is only formally correct near such points, and, as is well known, yields grossly inaccurate results for the pressure distribution near the ends of the body.

Using axial source distributions, and confining his interest to the flow in the immediate vicinity of the body, Van Dyke (1959) derived a formal second-order slender-body theory. As shown on the right half of figure 2, the theory yields surface pressure distributions $C_p(x)$ which diverge even more strongly near stagnation points than do the first-order results. Moreover, if, as in the case on which figure 2 is based (ellipsoid of revolution), the ends of the body are blunt, comparisons with exact solutions show the formal second-order theory to be in error by terms of order τ^4 even away from the stagnation points.

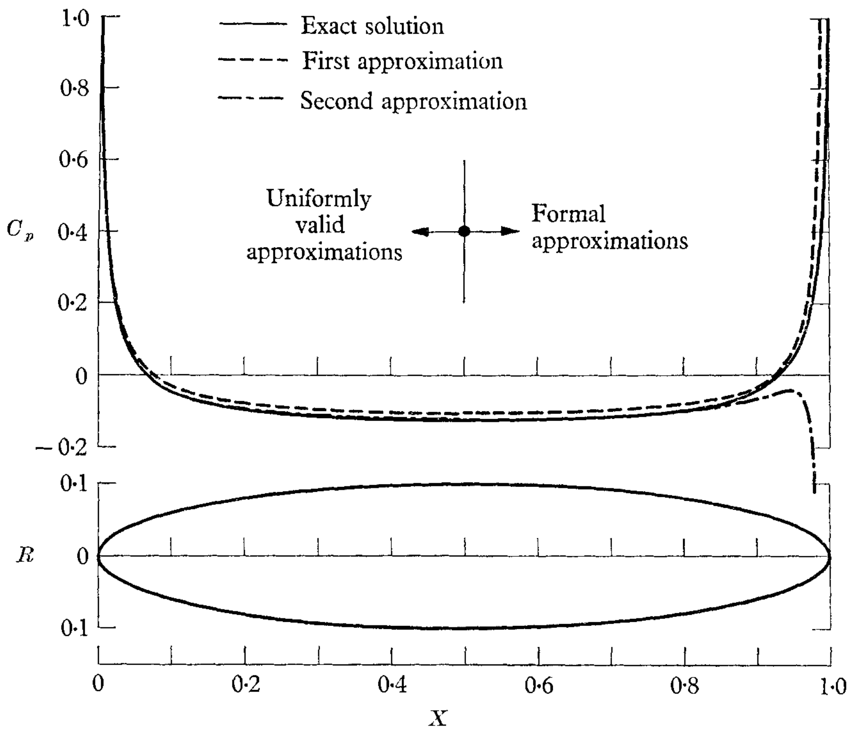

FIGURE 2. Comparison of various approximations to pressure distribution on prolate ellipsoid of revolution with exact solution. $\tau = 0{\cdot}2$.

It is of interest to compare the present uniformly valid results with those of formal slender-body theory. It can be shown (I) that the linearized result for the distribution of the axial velocity component along the body surface can be brought into agreement with our first-order result by subtracting the singular terms and multiplying the remainder by a corrective factor. This is highly reminiscent of Lighthill's (1951) rule for rendering thin-airfoil theory uniformly valid.

Of more fundamental importance is the comparison of the singularity strengths. As already noted, our first approximation for the source strength (3.28) is identical in form with the familiar slender-body formula. Surprisingly, we find that equation (3.33) also agrees in functional form with Van Dyke's (1959) formal second-order result. The only difference between the two distributions is in their extent. Whereas in formal slender-body theory the distribution is allowed to extend to the ends of the body, we have admitted the presence of gaps between the ends of the source distribution and the stagnation points, and have taken care to determine the correct extent of the gaps. Thus it appears that the errors of the formal theory, even in its second-order version, are due solely to the neglect of these gaps.

It is suspected that this conclusion may also hold outside the scope of the present research. We thus propose the following rule for rendering small-perturbation theory uniformly valid: before attempting to find the strength of the singularity distribution, predetermine its extent. To do so expand the governing integral equation in power series about the stagnation points, in the manner suggested by Flügge-Lotz (1931), developed by Landweber (1951), and followed here in § 3.1.

To be sure, the efficacy of such an approach would depend on the body shape; in the present case, we had to restrict ourselves to bodies with parabolic ends. However, within such limitations, we expect this rule to be applicable to any problem in which the linearized solution consists of singularities distributed along some mean surface or line. To support this claim, we note that Flügge-Lotz's (1931) suggestion was made in conjunction with the lifting problem for bodies of revolution. Moreover, Macagno (1962) has shown that the axial (two-dimensional) source distribution associated with an elliptic cylinder according to thin-airfoil theory does indeed generate a stagnation streamline in the shape of an ellipse, with foci at the ends of the distribution. Finally, our conjecture is consistent with Lighthill's (1951) interpretation of his thin-airfoil result: 'the velocity field obtained by a straight-forward expansion in powers of the disturbances...may be rendered a valid first approximation near the leading edge... if the whole field is shifted downstream parallel to the chord for a distance of half the leading edge radius of curvature.'

Van Dyke (1954, 1959) has derived a more heuristic scheme for rendering the predictions of slender-body theory, as to the body-surface velocity distribution, uniformly valid to second order. The extraneous singular terms were adjusted or removed simply by comparing the formal predictions with exact solutions for bodies which approximate the shape of the body under study near the stagnation points. If the body shape is given by equations (2.1), the appropriate comparison bodies are two ellipsoids of revolution, one having the same values of a_1 and a_2 as does the given body, and the other having the same b_1 and b_2.

Because of the complexity of the second approximations, we have been able to check our formulas with Van Dyke's only to first order (I). However, there is no doubt that the two sets of results are also equivalent in the second approximation, since our system for rendering slender-body theory uniformly valid to second order could have been derived by a procedure analogous with Van Dyke's. As noted above the present system is based on the introduction of gaps between the ends of the source distribution and the stagnation points. To a second approximation, α is from equation (3.15) the same for bodies described by equation (2.1) as it is for the ellipsoid having the same values of a_1 and a_2, and so could have been calculated to order τ^4 by examining the known exact solution for the ellipsoid.

4.3. *Comparisons with other solutions*

As implied above, we have checked our results with the exact closed-form solution for the ellipsoid of revolution. It is found (I) that our formulas for the extent of the source distribution, (3.15) and (3.16), and for the form of the distribution, (3.33), are simply the first few terms of the power-series expansions in τ^2 of the exact results. Similarly, our results for the surface pressure distribution check satisfactorily with the exact formulas. Numerical agreement between the second-order and exact results is, as can be seen from the left half of figure 2, excellent.

The rapid convergence of the successive approximations in the case of the ellipsoid does not hold for all body shapes satisfying the restrictions of § 2.1. We consider the profile defined by

$$R^2(x) = \tfrac{1}{4}\tau^2[1-(2x-1)^4]. \tag{4.1}$$

Landweber (1951, 1959) has studied the hydrodynamics of this body in great detail. It is somewhat blunter in appearance than is the ellipsoid, but not so blunt as is the Rankine ovoid.

The lengths of the gaps between the ends of the source distribution and the stagnation points are readily determined from equations (3.15) and (3.16)

$$\alpha = 1-\beta = \tfrac{1}{2}\tau^2+\tfrac{3}{4}\tau^4+\tfrac{11}{4}\tau^6+\tfrac{221}{16}\tau^8+\tfrac{1311}{16}\tau^{10}+O(\tau^{12}). \tag{4.2}$$

Since the first five coefficients of this expansion are positive, we suspect that subsequent coefficients are also positive. Then there is at least an upper limit on the thickness ratio for which equation (4.2) is applicable, since α and $1-\beta$ must be less than $\frac{1}{2}$. Moreover, examination of the coefficients of equation (4.2) suggests that the general term behaves like $n!\,\tau^{2n}$. This implies that (4.2) does not converge for any $\tau > 0$, but is only an asymptotic expansion in τ^2. Thus the example under study illustrates the possibility, noted in § 4.1, that the successive-approximation procedure developed in §§ 3.1 and 3.2 may not converge.

Explicit results for the source strength and for the surface pressure distribution are presented in I. One difficulty in computing these results is worth noting. The contribution of the logarithmic term in f_2, see equation (3.33), to the second-order pressure formula involves an integral which could not be reduced to closed form, but had to be evaluated numerically. Similar difficulties should be anticipated in analysing any body other than the ellipsoid or revolution, in which special case the argument of the logarithmic term in equation (3.33) is a constant.

The first and second approximations to the surface pressure distribution are compared in figure 3 with numerical results supplied by A. M. O. Smith and J. L. Hess of the Douglas Aircraft Company. Their computer programme (Smith & Pierce 1958) is based on the use of ring sources distributed over the body surface. The integral equation for the source strength is approximated by a finite number of algebraic equations, which they solve by a numerical iteration procedure. In the present case, they used successively increasing numbers of these equations—45, 90, and finally 180—and extrapolated the results so that the data on the C_p distribution could be guaranteed to $\pm 0{\cdot}0002$ over most of the body, and to $\pm 0{\cdot}002$ near the stagnation points. We may therefore regard their results as graphically exact, and use figure 3 to conclude that the second-order theory, while considerably more accurate than the first order, is still noticeably in error.

It is of interest to compare the performance of the successive-approximation procedure with that of various numerical methods developed prior to the Douglas computer programme. Landweber (1951) has computed the pressure

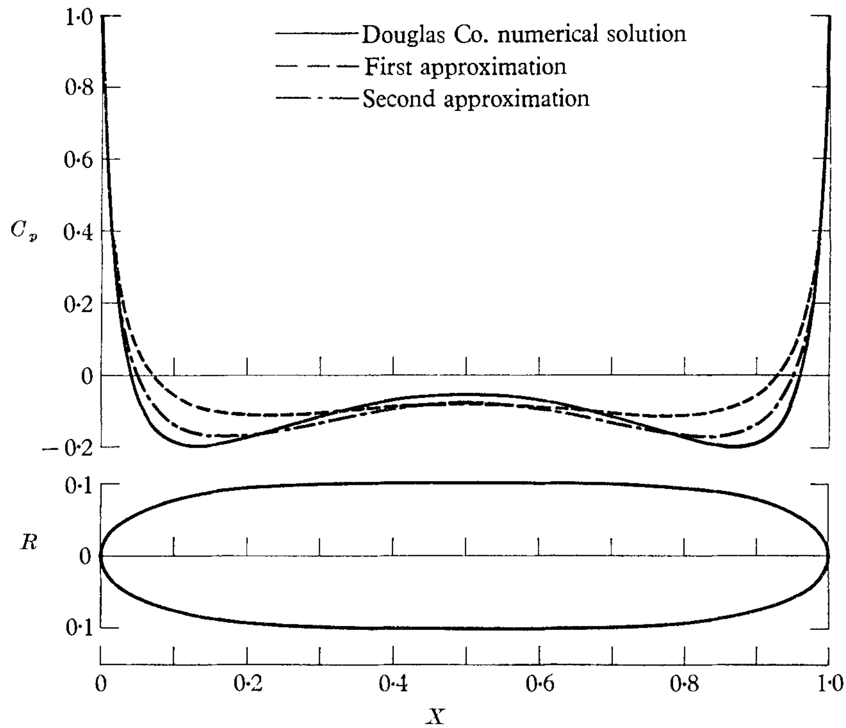

FIGURE 3. Comparison of first and second uniformly valid approximations to pressure distribution on Landweber's body (defined by equation (4.1)) with numerical solution. $\tau = 0{\cdot}2$.

distribution on the body defined by equation (4.1) with four different numerical procedures: one due to von Kármán (1927); one due to Kaplan (1935); and two due to Landweber (1951), one of which is based on axial doublet distribution, while the other uses ring vortices on the body surface. Both of Landweber's methods yield pressure distributions which are indistinguishable graphically from the Douglas results, except near the minimum C_p, where they differ by about 0·01. The von Kármán and Kaplan methods do not perform quite as well, but are still noticeably more accurate than is the uniformly valid second-order theory.

In view of the similarity between Landweber's axial-doublet-distribution method and the present procedure, see §§ 3.1 and 3.2, we may conclude that acceptable results could have been obtained with our procedure were it carried out beyond the second approximation, since Landweber's results are essentially a fourth-order approximation. This necessity for considering terms whose mathematical order of magnitude is quite small, but whose numerical magnitude is significant, is typical of asymptotic expansions.

These comparisons show that the successive-approximation procedure must be used with caution, and emphasize the need for a criterion for convergence. Although the present analysis has failed to provide such a criterion, it should be noted that lack of convergence was suspected for Landweber's body after finding the gaps between the ends of the source distribution and the stagnation points. Since we have available simple formulas for α and $(1-\beta)$ which enable a determination of the first five terms of the expansions of these quantities in τ^2, examination of such expressions might be generally useful for defining cases where the technique yields only an asymptotic expansion in τ^2.

4.4. *Interference problems*

We now extend the approach developed in §§ 3.1 and 3.2 to a class of unsteady interference problems. We still require the body to be shaped as stipulated in § 2.1, and the flow at infinity to be uniform. However, we now permit the presence of external disturbances moving with respect to the body. We require that these disturbances be axisymmetric, and that the singularities with which they are associated have a strength of order τ^2 and be located a distance large compared to τ^2 from the body under study.

It is convenient to work in terms of the stream function, which we split up as follows:

$$\Psi = \tfrac{1}{2}Ur^2 + \psi^{(b)} + \psi^{(e)}. \tag{4.3}$$

The first term is the stream function of a uniform flow, $\psi^{(b)}$ is the stream function of the body-bound axial source distribution, and $\psi^{(e)}$ is the stream function of the external singularities. The integral equation governing the body-bound source strength, which is now a function $f(x,t)$ of both location x and time t, is derived from equation (2.9) in the form

$$\int_\alpha^\beta f(\xi,t)\,K(x,\xi)\,d\xi = 2S(x) + \frac{4\pi}{U}\psi^{(e)}(x,R(x),t), \tag{4.4}$$

where the kernel $K(x,\xi)$ is defined in equation (2.7).

Within the above restrictions, the most general singularity involved in $\psi^{(e)}$ is a ring source of radius $r_0(t)$ and strength $Q(t) = O(\tau^2)$, situated in the plane $x = D(t)$, where

$$\{D^2+r_0^2\}^{-\frac{1}{2}} = O(1), \quad \{(D-1)^2+r_0^2\}^{-\frac{1}{2}} = O(1). \tag{4.5}$$

Since the x-axis is a streamline of the flow due to such a singularity, its stream function $\psi^{(e)}$ is constant on the axis. We may arbitrarily set this constant equal to zero, so that for small radial distances from the axis

$$\psi^{(e)}(x,r,t) \approx r\frac{\partial\psi^{(e)}}{\partial r}(x,0,t), \tag{4.6}$$

where we have retained only the first (largest) term of a Taylor-series expansion. But from equations (2.8), (4.6) may be written in terms of the potential $\phi^{(e)}$ as

$$\psi^{(e)}(x,r,t) \approx r^2\frac{\partial\phi^{(e)}}{\partial x}(x,0,t) \approx \frac{Q}{4\pi}\frac{r^2(x-D)}{[(x-D)^2+r_0^2]^{\frac{3}{2}}}, \tag{4.7}$$

where the result for the axial velocity due to the source ring is taken from Sadowsky & Sternberg (1950). Then, from equations (4.5) and (4.7), and the restriction that $Q = O(\tau^2)$

$$\psi^{(e)}(x, R(x), t) = O(\tau^4). \tag{4.8}$$

Since the equations (3.1)–(3.6) used to determine α are coefficients in the power-series expansion about the nose of the governing integral equation, and since we are able to ignore terms of order τ^4 in the solution of these equations for α, equations (4.4) and (4.8) show that the extent of the source distribution is not affected by the presence of the external singularities.

To determine the functional form of $f(x,t)$, we proceed exactly as in § 3.2. An nth approximation to the body-bound source strength is defined as in equation (3.25), with $f_0 = 0$. We also find it convenient to define an nth approximation to $\psi^{(e)}$, such that on the body surface

$$\psi^{(e)}(x, R(x), t) - \psi_n^{(e)}(x, R(x), t) = o(\tau^{2n+2}). \tag{4.9}$$

From equations (4.8) and (4.9), we may set $\psi_0^{(e)} = 0$.

Equation (4.4) may now be rewritten

$$\int_\alpha^\beta \left[f_n(\xi,t) - f_{n-1}(\xi,t)\right] K(x,\xi)\,d\xi = 2S(x) - \left[\int_\alpha^\beta f_{n-1}(\xi,t)\,K(x,\xi)\,d\xi + \frac{4\pi}{U}\psi_{n-1}^{(e)}\right]_{r=R(x)} + o(\tau^{2n}). \tag{4.10}$$

From equations (3.17) and (3.24) this has the solution

$$f_n(x,t) = S'(x) + f_{n-1}(x,t) - \frac{1}{2}\int_\alpha^\beta f_{n-1}(\xi,t)\,K_x(x,\xi)\,d\xi + \frac{2\pi}{U}\frac{\partial}{\partial x}[\psi_{n-1}^{(e)}(x, R(x), t)]. \tag{4.11}$$

Note that

$$f_1(x,t) = S'(x); \tag{4.12}$$

that is, to a first approximation, the body-bound source distribution is the same as if the body were isolated.

Equation (4.11) is well adapted to the usual interference problem, in which $\psi^{(e)}$ is not specified, but is to be determined such that $\psi^{(b)} + \psi^{(e)}$ satisfies some condition on a boundary of the flow other than the surface of the body under study, e.g. on the surface of a second body, or on a free surface. In such cases, one uses (4.12) to find $\psi_1^{(b)}$, determines $\psi_1^{(e)}$ such that $\psi_1^{(b)} + \psi_1^{(e)}$ satisfies the external boundary condition to a first approximation, plugs this result into (4.11) to find a second approximation for the strength of the body-bound source distribution, determines $\psi_2^{(e)}$ such that $\psi_2^{(b)} + \psi_2^{(e)}$ satisfy the external boundary condition to a second approximation, etc. Such a procedure has recently been employed (Moran & Kerney 1963) to find a solution for the vertical water-exit and -entry of a slender body of revolution which satisfies the boundary conditions on the body and on the free water surface to second order.

This research was supported by the Fluid Dynamics Branch of the Office of Naval Research, under contract Nonr-3396(00), Task NR 062-269. The author is pleased to acknowledge the encouragement and advice of his co-workers at

Therm Advanced Research, especially H. S. Tan, D. E. Ordway, and A. Ritter. He is also indebted to L. Landweber, whose work so strongly influenced the present analysis. Finally, the author expresses his appreciation to A. M. O. Smith and J. L. Hess, for their splendid co-operation in furnishing the numerical results shown in figure 3.

Appendix. Determination of second approximation to source strength

The function defined by equation (3.32) may be written,

$$H_k(x) = \int_{\alpha-x}^{\beta-x} \frac{s^k ds}{[s^2+R^2]^{\frac{1}{2}}}. \tag{A 1}$$

Adding and subtracting the quantity $s^{k-2}R^2$ to the numerator of the integrand of equation (A 1), and integrating by parts, we obtain the recursion formula

$$H_k(x) = \frac{1}{k}[(\beta-x)^{k-1}B(x)-(\alpha-x)^{k-1}A(x)]-\frac{k-1}{k}R^2(x)\,H_{k-2}(x) \quad (k=1,2,\ldots), \tag{A 2}$$

where

$$A(x) \equiv \{(x-\alpha)^2+R^2(x)\}^{\frac{1}{2}}, \tag{A 3}$$

$$B(x) \equiv \{(\beta-x)^2+R^2(x)\}^{\frac{1}{2}}. \tag{A 4}$$

From integral tables,

$$H_0(x) = L(x) \equiv \ln\left[\frac{\beta-x+B(x)}{\alpha-x+A(x)}\right]. \tag{A 5}$$

Then introducing equations (A 2) and (A 5) into (3.31), we obtain

$$J(x) = \tfrac{1}{2}B(x)\,F(x,\beta)-\tfrac{1}{2}A(x)\,F(x,\alpha)-\tfrac{1}{2}L(x)\,\{\tfrac{1}{2}R^2(x)\,S''(x)+O(\tau^6)\}, \tag{A 6}$$

where

$$F(x,y) = \sum_{k=1}^{\infty}\frac{1}{k!}(y-x)^{k-1}S^{(k)}(x) - R^2(x)\sum_{k=1}^{\infty}\frac{k+1}{k(k+2)!}(y-x)^{k-1}S^{(k+2)}(x)+O(\tau^6). \tag{A 7}$$

Note that the first term of equation (A 7) may be written

$$\sum_{k=1}^{\infty}\frac{1}{k!}(y-x)^{k-1}S^{(k)}(x) = \frac{S(y)-S(x)}{y-x}. \tag{A 8}$$

The terms A, B, and L may be simplified as follows. We rewrite (A 3) as

$$A(x) = \{(x+\alpha)^2+[R^2-4\alpha x]\}^{\frac{1}{2}}. \tag{A 9}$$

It is readily verified that the term in square brackets is always smaller in order of magnitude than $(x+\alpha)^2$ by a factor of τ^2. When $x \gg \alpha$ this is obvious, while when $x = O(\tau^2)$, we have, from equations (2.1) and (3.15),

$$R^2-4\alpha x = a_2x^2+\tfrac{1}{4}a_1a_2x+O(\tau^8) = a_2x(x+\alpha)+O(\tau^8) = \frac{x+\alpha}{x}[R^2-a_1x]+O(\tau^8). \tag{A 10}$$

The last expression is also a valid first approximation when $x \gg \alpha$, in which case it is accurate to order τ^2.

Thus if we factor out $(x+\alpha)$ from (A 9), the radical is of the form $1+O(\tau^2)$, and may be expanded accordingly,

$$A(x) = [x+\alpha]+\frac{1}{2}\left[\frac{R^2-a_1x}{x}\right]+O(\tau^4x). \tag{A 11}$$

Similarly,

$$B(x) = [(1-x)+(1-\beta)]+\frac{1}{2}\left[\frac{R^2-b_1(1-x)}{1-x}\right]+O(\tau^4[1-x]). \tag{A 12}$$

Now turning to L we rewrite equation (A 5) as

$$L(x) = \ln\left[\left(\frac{\beta-x+B}{\alpha-x+A}\right)\left(\frac{x-\alpha+A}{x-\alpha+A}\right)\right]$$

$$= \ln\left[\frac{\{A+x-\alpha\}\{B+(1-x)-(1-\beta)\}}{R^2}\right], \tag{A 13}$$

where we have made use of (A 3). From (A 11)

$$\ln\{A+x-\alpha\} = \ln\left[2x+\frac{1}{2}\frac{R^2-a_1x}{x}+O(\tau^4x)\right]$$

$$= \ln 2x+\frac{1}{4}\frac{R^2-a_1x}{x^2}+O(\tau^4). \tag{A 14}$$

A similar relation may be found for the term in (A 13) involving B. Thus we find

$$L(x) = \ln\left[\frac{4x(1-x)}{R^2(x)}\right]+\frac{1}{4}\frac{R^2-a_1x}{x^2}+\frac{1}{4}\frac{R^2-b_1(1-x)}{(1-x)^2}+O(\tau^4). \tag{A 15}$$

Equation (A 11) may be further simplified to

$$A(x) = x-\alpha+\frac{1}{2x}R^2(x)+O(\tau^4x). \tag{A 16}$$

In similar fashion we find

$$\frac{R^2(\alpha)-R^2(x)}{\alpha-x} = \frac{R^2(x)}{x}+O(\tau^4). \tag{A 17}$$

Then, from equations (A 7), (A 8), (A 16), and (A 17)

$$A(x)\,F(x,\alpha) = S(x)-S(\alpha)+\frac{1}{2}\left[\frac{R^2(x)}{x}\right]^2$$

$$+R^2(x)\sum_{k=1}^{\infty}\frac{k+1}{k(k+2)!}(-x)^k S^{(k+2)}(x)+O(\tau^6). \tag{A 18}$$

The contribution of the B-term to (A 6) is of similar form, while that of the L-term is quite easily computed with the aid of (A 15). Substituting the resultant relation for J into equation (3.29), we obtain the formula for f_2 given in the text as (3.33).

REFERENCES

FLÜGGE-LOTZ, I. 1931 Calculations of potential flow past airship bodies in yaw. *Ingenieur-Archiv*, **2**, 507. (Translated by J. Vanier, 1931, *NACA TM* 675.)

KAPLAN, C. 1935 Potential flow about elongated bodies of revolution. *Nat. Adv. Comm. Aero., Wash., Rept.* no. 516.

LANDWEBER, L. 1951 The axially symmetric potential flow about elongated bodies of revolution. *David W. Taylor Model Basin, Rept.* no. 761.

LANDWEBER, L. 1959 Potential flow about bodies of revolution and symmetric two-dimensional forms. *Iowa Inst. of Hydraulic Res., State U. of Iowa, Rept.*

LIGHTHILL, M. J. 1951 A new approach to thin-airfoil theory. *Aero. Quart.* **3**, 193.

MACAGNO, M. 1962 Image system in an ellipse due to an external source. *Iowa Inst. of Hydraulic Res., State U. of Iowa, Rept.*

MORAN, J. P. 1961 Line source distributions in axisymmetric incompressible flow. I. Conditions for applicability. *Therm Adv. Res. Tech. Rept.* no. 616.

MORAN, J. P. 1962 (referred to as I) Line source distributions in axisymmetric incompressible flow. III. Solution of the direct problem. *Therm. Adv. Res. Tech. Rept.* no. 622.

MORAN, J. P. & KERNEY, K. P. 1963 On the small-perturbation theory of water-exit and -entry. *Proc. 8th Midwestern Mech. Conf.* (to be published).

MUNK, M. M. 1924 The aerodynamic forces on airship hulls. *Nat. Adv. Comm. Aero., Wash., Rept.* no. 184.

MUNK, M. M. 1934 Fluid mechanics, part II. *Aerodynamic Theory* (ed. W. F. Durand), Vol. I, Division D. Berlin: Julius Springer.

SADOWSKY, M. A. & STERNBERG, E. 1950 Elliptic integral representation of axially symmetric flows. *Quart. Appl. Math.* **8**, 113.

SMITH, A. M. O. & PIERCE, J. 1958 Exact solution of the Neumann problem. Calculation of non-circulatory plane and axially symmetric flows about and within arbitrary boundaries. *Douglas Aircraft Co., Rept.* no. ES 26988.

VANDREY, F. 1951 A direct iteration method for the calculation of the velocity distribution of bodies of revolution and symmetrical profiles. *Admiralty Res. Lab., Teddington, Rept.* no. RI/G/HY/12/2.

VAN DYKE, M. D. 1954 Subsonic edges in thin-wing and slender-body theory. *Nat. Adv. Comm. Aero., Wash., Tech. Note* 3343.

VAN DYKE, M. D. 1959 Second-order slender-body theory—axisymmetric flow. *Nat. Aero. Space Adm., Wash., Rept.* no. R-47.

VON KÁRMÁN, T. 1927 Calculation of pressure distribution on airship hulls. *Abhand. Aero. Inst. Tech. Hoch. Aachen*, no. 6, 3 (translated by D. W. Miner, 1930, *NACA TM* 574).

WEINIG, F. 1928 Über schnell konvergierende graphische lösungen von stromungsproblemen durch integralgleichungen. *Z. tech. Phys.* **9**, 39.